Exercise 58

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4]$$

Solution

Take the derivative of the function.

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(t + \cot\frac{t}{2}\right) \\ &= 1 + \left(-\csc^2\frac{t}{2}\right)\cdot\frac{d}{dt}\left(\frac{t}{2}\right) \\ &= 1 + \left(-\csc^2\frac{t}{2}\right)\cdot\left(\frac{1}{2}\right) \\ &= 1 - \frac{1}{2}\csc^2\frac{t}{2} \end{aligned}$$

Set $f'(t) = 0$ and solve for t.

$$1 - \frac{1}{2}\csc^2\frac{t}{2} = 0$$

$$\csc^2\frac{t}{2} = 2$$

$$\frac{1}{\sin^2\frac{t}{2}} = 2$$

$$\sin^2\frac{t}{2} = \frac{1}{2}$$

$$\sin\frac{t}{2} = \pm\frac{1}{\sqrt{2}}$$

$$\frac{t}{2} = \frac{\pi}{4} + \frac{\pi}{2}n, \quad n = 0, \pm1, \pm2, \ldots$$

$$t = \frac{\pi}{2} + n\pi, \quad n = 0, \pm1, \pm2, \ldots$$

Only $t = \pi/2$ and $t = 3\pi/2$ are within $[\pi/4, 7\pi/4]$, so evaluate f at these values.

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cot\frac{\pi}{4} = \frac{\pi}{2} + 1 \approx 2.5708 \qquad \text{(absolute minimum)}$$

$$f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} + \cot\frac{3\pi}{4} = \frac{3\pi}{2} - 1 \approx 3.71239 \qquad \text{(absolute maximum)}$$

Now evaluate the function at the endpoints of the interval.

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \cot\frac{\pi}{8} \approx 3.19961$$

$$f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} + \cot\frac{7\pi}{8} = \frac{3\pi}{2} - 1 \approx 3.08357$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $[\pi/4, 7\pi/4]$. The graph of the function below illustrates these results.